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On the transversal vibrations of an axially moving continuum with a time-varying velocity: Transient from string to beam behavior

S.V. Ponomareva *, W.T. van Horssen

Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft Institute of Applied Mathematics, Mekelweg 4, 2628 CD Delft, The Netherlands

A R T I C L E   I N F O

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A B S T R A C T

In this paper an initial boundary value problem for a linear equation describing an axially moving stretched beam will be considered. The velocity of the beam is assumed to be time varying. Since the order of magnitude of the bending stiffness terms depends on the vibration modes and the frequencies involved a combination of two simplified models (that is, a string equation for the lower frequencies and a beam with string effect equation for the higher frequencies) will be used to describe the transversal vibrations of the system accurately. Based on the calculations of the natural frequencies the regions of applicability of these sub-models will be determined. A two time-scales perturbation method will be used to construct formal asymptotic approximations of the solutions. Non-resonant and some resonant cases will be studied for four different values of the relative errors. An important implication of the earlier results in the literature is that for these types of axially moving continua problems the use of only string-like models is not appropriate. To describe the dynamics of these types of axially moving continua problems correctly one has to include (small) bending stiffness in the model. In this paper it is explicitly shown how to work with a combined model that is a string model at the low frequencies and a tensioned beam model at the higher frequencies.

1. Introduction

Axially moving systems are present in a vast class of engineering problems which arise in industrial, civil, aerospatial, mechanical, electronic, medical, and automotive applications. Serpentine belts, aerial cables, tram and train ways, oil pipelines, magnetic tapes, power transmission belts, band saw blades, chair lifts in skiing resorts, and even models of human DNA are examples of real objects where axial transport of mass can be associated with transverse vibrations. Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years by many researches and still is of interest today (see the references for a recent overview). In the classical analysis of axially moving continua the vibrations are usually classified into two types, i.e. whether it is of a string-like type or of a beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibrations of a belt system (with time-varying velocity $V(T)$) can be modeled mathematically as:

* Corresponding author.
E-mail addresses: S.Ponomareva@tudelf.nl (S.V. Ponomareva), W.T.vanHorssen@tudelft.nl (W.T. van Horssen).
URL: http://www.tudelft.nl (S.V. Ponomareva).

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- string-like by

$$U_{TT} + 2VU_{XT} + V_T U_X + (V^2 - c^2)U_{XX} = 0 \tag{1}$$

and

- beam-like (with a string effect) by

$$U_{TT} + 2VU_{XT} + V_T U_X + (V^2 - c^2)U_{XX} + \frac{EI}{\rho A}U_{XXXX} = 0, \tag{2}$$

where A is the area of the cross-section of the belt, X the coordinate in horizontal direction, $U(X,T)$ the displacement of the string in vertical direction, πL the distance between the pulleys, ρ the mass density of the belt, E the modulus of elasticity, I the moment of inertia with respect to the x (horizontal)-axis, T the time, $V(T)$ the time-varying belt speed, and c the wave speed and where $c = \sqrt{T_0/\rho A}$, in which T_0 is assumed to be the constant tension of the belt. The time-varying belt velocity $V(T)$ is given by $V(T) = \varepsilon(\overline{V}_0 + \overline{\alpha}\sin(\overline{\omega}T))$, where $\overline{V}_0$, $\overline{\omega}$, and $\overline{\alpha}$ are some positive constants with $\overline{V}_0 > 0$ and $\overline{V}_0 > |\overline{\alpha}|$, and where ε is a small parameter with $0 < \varepsilon \ll 1$. The term $\varepsilon\overline{\alpha}\sin(\overline{\omega}T)$can be seen as a small perturbation of the main belt speed $\varepsilon\overline{V}_0$, due to different kinds of imperfections of the belt system. The small parameter ε indicates that the belt speed $V(T)$ is small compared to the wave speed c. The condition $\overline{V}_0 > |\overline{\alpha}|$ guarantees that the belt always moves forward in one direction.

Due to different kinds of imperfections of the belt system such as roll eccentricities and varying belt speed, severe transversal vibrations (due to internal resonances) can occur. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support components, and an increase of energy consumption of the belt system. By knowing the natural frequencies of the belt, the so-called resonance-free belt system can be designed. Although the nonlinear models can be more informative, and describe the real conveyer belt systems usually better, it is not meaningless to investigate the linear equations (1) and (2) first.

Eq. (1) with $V(T) = \varepsilon(\overline{V}_0 + \overline{\alpha}\sin(\overline{\omega}T))$ was studied in Ref. [1]. It was found that there are infinitely many values of $\overline{\omega}$ giving rise to internal resonances in the belt system. It was also shown that the truncation method cannot be applied in order to obtain asymptotic results on long time-scales (that is, on time-scales of order ε^{-1}). Eq. (1) with $V(T) = \overline{V}_0 + \varepsilon\overline{\alpha}\sin(\overline{\omega}T)$ was studied in Ref. [2]. It was shown that in this case for a high velocity the truncation method also cannot be applied. On the other hand, it was also shown in Ref. [3] that for the beam equation (2) the truncation method can be applied, but the dynamic behavior of the belt system is still very complicated. The stability conditions for the belt system were also derived in Ref. [3]. From experiments and from the theoretical investigations (see, for example, Ref. [4]) it is known that the real dynamic behavior of conveyer belt systems with relatively small bending stiffness is some sort of combination of both models (1)–(2). The low frequency vibration modes look more like string modes and the higher order modes (when the bending stiffness becomes more important) look more like beam modes. It is not only interesting but also important from the applicational point ofview to investigate such phenomena of transients "from string to beam" behavior. In Ref. [4] the effect of bending stiffness on higher order modes was discussed. In recent papers [5–7] the following attempts to describe these phenomena can be found. When the belt speed is high and has the same order of magnitude as the wave speed c (that is, $V(T) = \overline{V}_0 + \varepsilon\overline{\alpha}\sin(\overline{\omega}T)$) a case has been studied in Ref. [5] for which the bending stiffness is of order ε, and an approximate analytical expression for the natural frequency and stability regions has been presented. In Ref. [6] the same assumptions have been used and boundary layer solutions have been constructed. Approximations of the eigenvalues of the belt system with a small bending stiffness were also presented in Ref. [7]. All the authors of the aforementioned papers found that the natural frequencies change due to the presence of a small bending stiffness, but did not include the fact that for the higher order modes the bending stiffness terms are not of order ε anymore, and actually should be included in the $\mathcal{O}(1)$-problem. Namely, the assumption that most axially moving belt systems have small bending stiffness relative to their tension and can be modeled as an axially moving beam with small dimensionless bending stiffness (as it was done in Ref. [7]) is only valid for lower order vibration modes. Moreover, the natural frequencies of the beam model (2) with $V(T) = \overline{V}_0 + \varepsilon\overline{\alpha}\sin(\overline{\omega}T)$ cannot be found exactly (see for instance Refs. [10,11]). In this paper it will be assumed that$V(t) = \varepsilon(\overline{V}_0 + \overline{\alpha}\sin(\overline{\omega}T))$. The idea how and when in this case different simplified models may be applied to construct a more realistic model of the traveling belt system was proposed in Ref. [12]. Usually it is not possible to calculate the natural frequencies of a real belt system exactly. The bending stiffness, however, is not important for the lower modes of vibrations. And for the higher modes of vibration the bending stiffness terms become more important than the string terms.

Let us assume that $\overline{\omega}$ is given, and that initially at $T = 0$ the first N modes are present. If N is not too large the bending stiffness terms in the problem equation are small, and can be neglected. A string equation (see Eq. (1)) is then obtained. The problem for the string equation, however, has a serious drawback: the truncation method cannot be applied due to the presence of internal resonances for which all modes are interacting. And as a consequence, no good approximations of the exact solution on a long time-scale can be found. Moreover, the resonance frequencies (which are found in this way) might not correspond to the exact resonance frequencies. Now, there are at least three simplified models depending on the vibration modes and the corresponding frequencies: a string model for the lower frequencies, a beam–string model (that is, the exact model) for the intermediate frequencies, and a pure beam model for the higher frequencies. A combination of these models can improve the results of the existing models and methods. The proposed method is based on calculating the natural frequencies of each sub-model, and determining the relative errors in it. In this way one can define intervals of applicability of these simplified models with a predefined, desired accuracy. It should be observed that a pure beam equation cannot be truncated either (see Ref. [13]), whereas for a beam–string model there is no problem with truncation. In this paper a combination of two simplified models will be proposed: a string model for the lower frequencies, and a string–beam model for the intermediate and the higher frequencies. It will turn out that this combination model has the following advantages: (1) The model allows to truncate a string equation correctly. (2) The natural frequencies of the lower vibration modes can be found more accurately, as the small bending stiffness effect has to be included as a small perturbation into the problem.

When a string-like model is used it has been shown in Refs. [1,2] that the infinite series representation for the solution cannot be truncated from the mathematical point of view (else one can obtain wrong internal resonances, and incorrect modal interactions). So, using only a string-like model for which the number of modes is truncated to a finite number can or will lead to wrong mathematical results. On the other hand, an infinite mode representation for the solution of a string-like model is physically irrelevant since for the higher order modes the bending stiffness becomes important, and so the beam–string-like equation has to be studied. In this paper it will be demonstrated how to work with the combined model that is a string model at the low frequencies and a tensioned beam model at the higher frequencies, such that mathematically and physically correct results are obtained.

The paper is organized as follows. In Section 2 the formulation of the problem will be given. The regions of applicability of the simplified models will be determined. In Section 3 the two time-scales perturbation method will be applied to construct approximate solutions of the problems. Values of $\overline{\omega}$ that give rise to internal resonances will be presented. The non-resonant case and some resonant cases will be studied for four different values of the relative error in Sections 4 and 5. Stability properties of the solution will also be given in Section 5. Finally, in Section 6 some conclusions will be drawn and some remarks will be made.

2. Formulation of the problem

In this section a new approach will be proposed to construct a model for an axially moving continuum, which includes both string type and beam type dynamic behavior. The simplest mechanical model for a traveling belt is a simply supported tensioned Euler–Bernoulli beam (see Fig. 1). The equation for this model is given by (see also Eq. (2))

$$U_{TT} + 2VU_{XT} + V_T U_X + (V^2 - c^2)U_{XX} + \frac{EI}{\rho A} U_{XXXX} = 0. \tag{3}$$

The speed of the belt is assumed to be time-varying and to be given by $V(T) = \varepsilon(\overline{V}_0 + \overline{\alpha}\sin(\overline{\omega}T))$. The boundary conditions and the initial conditions for Eq. (3) are given by

$$U(0,T;\varepsilon) = U(\pi L,T;\varepsilon) = U_{XX}(0,T;\varepsilon) = U_{XX}(\pi L,T;\varepsilon) = 0, \quad T \geqslant 0,$$

$$U(X,0;\varepsilon) = \overline{f}(X) \quad \text{and} \quad U_T(X,0;\varepsilon) = \overline{r}(X), \;\; 0 < X < \pi L, \tag{4}$$

where $\overline{f}(X)$ represents the initial displacement of the belt, $\overline{r}(X)$ is the initial velocity of the belt, and where πL is the distance between the pulleys. For simplicity it is assumed that the cross-section of the belt has a rectangular shape, so that $A = hb$ and $I = bh^3/12$, where h is the thickness and b is the width of the belt cross-section, respectively (see Fig. 2). Following the 3D theory of elasticity additional conditions have to be imposed to the stretched beam equation (3), that is: $\pi L/k \gg h$ and $b \gg h$, where k is the mode number.

Eq. (3) in non-dimensional form becomes

$$u_{tt} - u_{xx} + \mu u_{xxxx} = \varepsilon(-\alpha\omega\cos(\omega t)u_x - 2(V_0 + \alpha\sin(\omega t))u_{xt}) - \varepsilon^2(V_0 + \alpha\sin(\omega t))^2 u_{xx}, \tag{5}$$

where $x = X/L$, $V_0 = \overline{V}_0/c$, $t = (c/L)T$, $u = U/L$, $\omega = (L/c)\overline{\omega}$, $\alpha = \overline{\alpha}/c$ and $\mu = EI/\rho Ac^2L^2 = Eh^2/12\rho c^2L^2$. The boundary conditions and the initial conditions for Eq. (5) are given by

$$u(0,t;\varepsilon) = u(\pi,t;\varepsilon) = u_{xx}(0,t;\varepsilon) = u_{xx}(\pi,t;\varepsilon) = 0, \quad t \geqslant 0, \tag{6}$$

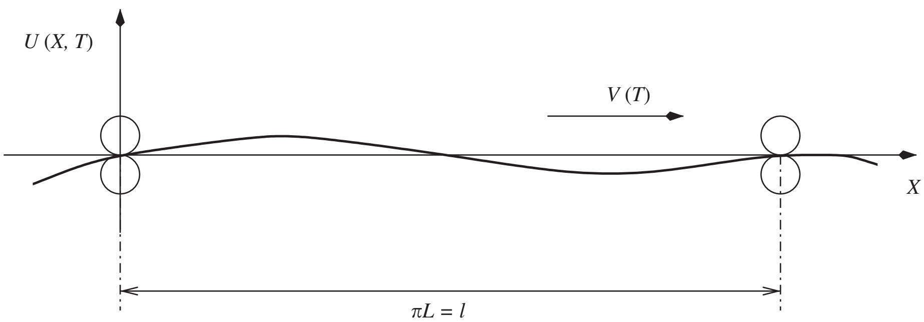

Fig. 1. The traveling belt system.

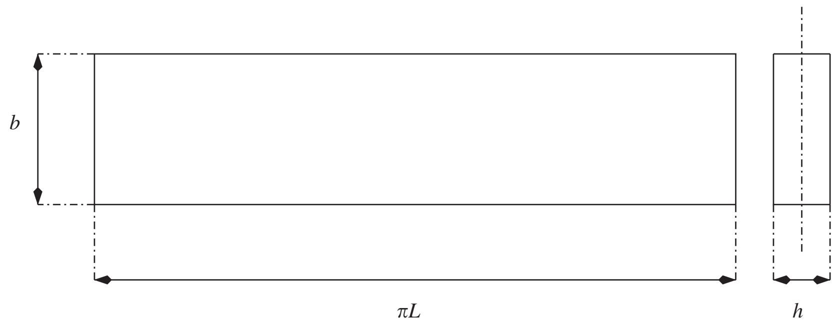

Fig. 2. The traveling belt configuration.

$$u(x,0;\varepsilon)=f(x) \quad \text{and} \quad u_t(x,0;\varepsilon)=r(x), \quad 0<x<\pi, \tag{7}$$

where $f(x)=\bar{f}(X)/L$ and $r(x)=\bar{r}(X)/c$.

As it was explained in the Introduction, the natural frequencies can usually not be calculated exactly for real problems of traveling belts due to the presence of complicated boundary conditions, different sorts of (non-)linearities, variable stiffness, and so on. It is not meaningless to consider only string behavior for the lower vibration modes of the belt (when the influence of bending stiffness is very small and can be neglected in the $\mathcal{O}(1)$-problem), and the following approach shows how one can define the regions of applicability. Let us first consider the equation:

$$u_{tt}-u_{xx}+\mu u_{xxxx}=0, \tag{8}$$

subjected to the boundary conditions (6). The parameter μ (as defined in Eq. (5)) is usually a small parameter for belt systems. To determine the natural frequencies of this problem the method of separation of variables can be used, giving as non-trivial solutions for $k=1,2,3,\ldots$:

$$e^{i\Omega_k t}\sin(kx), \tag{9}$$

where $i=\sqrt{-1}$, and

$$\Omega_k=k\sqrt{1+\mu k^2}. \tag{10}$$

Eq. (10) gives us exact natural frequencies for Eq. (8) subjected to the boundary conditions (6). For the string model (i.e. Eq. (8) without bending stiffness) and for the beam model (i.e. Eq. (8) without string effect) the natural frequencies also can be found, so that

$$\Omega_k^{(1)}=k \quad \text{for the string model,}$$

$$\Omega_k^{(2)}=k\sqrt{1+\mu k^2} \quad \text{for the stretched beam model, and}$$

$$\Omega_k^{(3)}=k^2\sqrt{\mu} \quad \text{for the beam model.} \tag{11}$$

It is possible now to find intervals of applicability of these simplified models (for k), based on the natural frequencies (11), with a desired or required accuracy. In Table 1 these regions for k are given (where the simplified models, i.e. the string model and the beam model, can be used) for μ equal to 0.0001, 0.002, 0.01, and 0.1, and with relative errors of at most 0.1%, 1%, 3%, and 5% in the frequencies, respectively.

Let us consider a real moving belt, fabricated from rubber, with the following mechanical properties: $E=1.8\,\text{GPa}$, $\rho=1.5\,\text{g}\,\text{cm}^{-3}$, $h=0.8\,\text{cm}$, $l=100\,\text{m}$, $T_0=5\,\text{N}\,\text{mm}^{-1}$, and $b<3000\,\text{mm}$. This implies that $\mu=0.002$. From Table 1 it can be seen that with a relative error of 5% in the frequencies the following model can be derived (the original initial-boundary value problem for Eq. (5) can now be split up by assuming that $u(x,t)=\Sigma_{k=1}^{\infty}u_k(t)\sin(kx)$):

For $1\leqslant k\leqslant 7$—*the string model*:

$$u_{tt}-u_{xx}=\varepsilon(-\alpha\omega\cos(\omega t)u_x-2(V_0+\alpha\sin(\omega t))u_{xt}-c_1u_{xxxx})-\varepsilon^2(V_0+\alpha\sin(\omega t))^2u_{xx}, \tag{12}$$

where it is assumed in Eq. (12) that $\mu u_{xxxx}=\mathcal{O}(\varepsilon)$, so that $\mu u_{xxxx}=\varepsilon c_1u_{xxxx}$.

For $8\leqslant k\leqslant 68$—*the beam with string effect model*:

$$u_{tt}-u_{xx}+\mu u_{xxxx}=\varepsilon(-\alpha\omega\cos(\omega t)u_x-2(V_0+\alpha\sin(\omega t))u_{xt})-\varepsilon^2(V_0+\alpha\sin(\omega t))^2u_{xx}, \tag{13}$$

where it should be observed that the terms in the left-hand side of Eq. (13) are of leading order, and are of the same order of magnitude.

Table 1
Applicability regions for the simplified models: string, beam with string effect, and beam equations.

Rel. error; model	μ			
	0.0001	0.002	0.01	0.1
String	$1\leqslant k\leqslant 4$	$k=1$	–	–
0.1%; string–beam	$5\leqslant k\leqslant 2234$	$2\leqslant k\leqslant 691$	$1\leqslant k\leqslant 223$	$1\leqslant k\leqslant 70$
Beam	$2235\leqslant k<\infty$	$692\leqslant k<\infty$	$224\leqslant k<\infty$	$71\leqslant k<\infty$
String	$1\leqslant k\leqslant 14$	$1\leqslant k\leqslant 3$	$k=1$	–
1%; string–beam	$15\leqslant k\leqslant 702$	$4\leqslant k\leqslant 161$	$2\leqslant k\leqslant 70$	$1\leqslant k\leqslant 23$
Beam	$703\leqslant k<\infty$	$162\leqslant k<\infty$	$71\leqslant k<\infty$	$24\leqslant k<\infty$
String	$1\leqslant k\leqslant 25$	$1\leqslant k\leqslant 5$	$1\leqslant k\leqslant 2$	–
3%; string–beam	$26\leqslant k\leqslant 399$	$6\leqslant k\leqslant 89$	$3\leqslant k\leqslant 39$	$1\leqslant k\leqslant 12$
Beam	$400\leqslant k<\infty$	$90\leqslant k<\infty$	$40\leqslant k<\infty$	$13\leqslant k<\infty$
String	$1\leqslant k\leqslant 32$	$1\leqslant k\leqslant 7$	$1\leqslant k\leqslant 3$	$k=1$
5%; string–beam	$33\leqslant k\leqslant 304$	$8\leqslant k\leqslant 68$	$4\leqslant k\leqslant 30$	$2\leqslant k\leqslant 9$
Beam	$305\leqslant k<\infty$	$69\leqslant k<\infty$	$31\leqslant k<\infty$	$10\leqslant k<\infty$

For $69\leqslant k<\infty$—*the beam model*:

$$u_{tt}+\mu u_{xxxx}=\varepsilon\left(-\alpha\omega\cos(\omega t)u_x-2(V_0+\alpha\sin(\omega t))u_{xt}+\frac{1}{\varepsilon}u_{xx}\right)-\varepsilon^2(V_0+\alpha\sin(\omega t))^2u_{xx}, \tag{14}$$

where it should be observed that the terms in the left-hand side of Eq. (14) are at least an order of magnitude larger than those terms in the right-hand side of Eq. (14).

As it was shown in Ref. [1] the truncation method cannot be applied to the string equation (12), but for the beam with string effect equation (13) the method can be applied (see Ref. [3]) when the internal resonances are taken into account. In Ref. [13] it was shown that the truncation method also cannot be applied to the beam equation (14) when one wants to obtain accurate approximations of the solution on long time-scale. Based on these observations it is assumed now for simplicity that the original problem (5) is split up into two models: for $1\leqslant k\leqslant 7$—the string model (12), and for $8\leqslant k<\infty$—the beam with string effect model (13). It was shown in Ref. [3] that by substituting $u(x,t)=\sum_{n=1}^{\infty}u_n(t;\varepsilon)\sin(nx)$ into Eq. (5), by multiplying both sides of the so-obtained equation with $\sin(kx)$, and then by integrating with respect to x from $x=0$ to $x=\pi$ it follows that

$$\ddot{u}_k+(\mu k^4+k^2)u_k=\varepsilon\sum_{n=1}^{\infty*}\frac{kn}{(n^2-k^2)\pi}(4\alpha\omega\cos(\omega t)u_n+8(V_0+\alpha\sin(\omega t))\dot{u}_n)+\mathcal{O}(\varepsilon^2), \tag{15}$$

where the $*$ in $\sum_{n=1}^{\infty*}$ indicates that the summation is only carried out for $n\pm k$ is odd. For $t=0$ $u_k(t)$ satisfies: $u_k(0;\varepsilon)=2/L\pi\int_0^{\pi}f(x)\sin(kx)\,dx$, and $\dot{u}_k(0;\varepsilon)=2/c\pi\int_0^{\pi}r(x)\sin(kx)\,dx$. In the next section a two time-scales perturbation method will be applied to approximate the solution of Eq. (15).

3. Application of the two time-scales perturbation method

To avoid secular terms in the approximate solution of Eqs. (5) and (15) a two time-scales perturbation method is used. The two new time scales are $t_0=t$ and $t_1=\varepsilon t$, implying that $u_k(t;\varepsilon)=v_k(t_0,t_1;\varepsilon)$. The following transformations are needed for the time derivatives:

$$\frac{du_k}{dt}=\frac{\partial v_k}{\partial t_0}+\varepsilon\frac{\partial v_k}{\partial t_1},$$

$$\frac{d^2u_k}{dt^2}=\frac{\partial^2 v_k}{\partial t_0^2}+2\varepsilon\frac{\partial^2 v_k}{\partial t_0\partial t_1}+\varepsilon^2\frac{\partial^2 v_k}{\partial t_1^2}. \tag{16}$$

It is assumed that $v_k=v_{k0}+\varepsilon v_{k1}+\cdots$, and $v_k(t_0,t_1;\varepsilon)=v_k^{(1)}(t_0,t_1;\varepsilon)$ for $1\leqslant k\leqslant 7$ (i.e. the time behavior is mainly determined by the string model), and $v_k(t_0,t_1;\varepsilon)=v_k^{(2)}(t_0,t_1;\varepsilon)$ for $8\leqslant k<\infty$ (i.e. the time behavior is mainly determined by the beam with string effect model). So that, in fact there are two sets of $\mathcal{O}(1)$ problems, two sets of $\mathcal{O}(\varepsilon)$ problems, and so on. By substituting Eq. (16) into Eq. (15), by substituting the expansion for v_k into Eq. (15), and by taking together terms of equal powers in ε for $v_k^{(1)}$ and $v_k^{(2)}$ it follows that for $1\leqslant k\leqslant 7$:

$$\mathcal{O}(1)^{(1)}:\frac{\partial^2 v_{k0}^{(1)}}{\partial t_0^2}+k^2 v_{k0}^{(1)}=0,$$

$$\begin{aligned}\mathcal{O}(\varepsilon)^{(1)}:\frac{\partial^2 v_{k1}^{(1)}}{\partial t_0^2}+k^2 v_{k1}^{(1)}=&-2\frac{\partial^2 v_{k0}^{(1)}}{\partial t_0\partial t_1}-c_1k^4v_{k0}^{(1)}+\sum_{n=1}^{7*}\left(\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)v_{n0}^{(1)}+8(V_0+\alpha\sin(\omega t))\frac{\partial v_{n0}^{(1)}}{\partial t_0}\right)\\&+\sum_{n=8}^{\infty*}\left(\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)v_{n0}^{(2)}+8(V_0+\alpha\sin(\omega t))\frac{\partial v_{n0}^{(2)}}{\partial t_0}\right),\end{aligned}\tag{17}$$

and for $8\leqslant k<\infty$:

$$\mathcal{O}(1)^{(2)}:\frac{\partial^2 v_{k0}^{(2)}}{\partial t_0^2}+(k^2+\mu k^4)v_{k0}^{(2)}=0,$$

$$\begin{aligned}\mathcal{O}(\varepsilon)^{(2)}:\frac{\partial^2 v_{k1}^{(2)}}{\partial t_0^2}+(k^2+\mu k^4)v_{k1}^{(2)}=&-2\frac{\partial^2 v_{k0}^{(2)}}{\partial t_0\partial t_1}+\sum_{n=1}^{7*}\left(\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)v_{n0}^{(1)}+8(V_0+\alpha\sin(\omega t))\frac{\partial v_{n0}^{(1)}}{\partial t_0}\right)\\&+\sum_{n=8}^{\infty*}\left(\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)v_{n0}^{(2)}+8(V_0+\alpha\sin(\omega t))\frac{\partial v_{n0}^{(2)}}{\partial t_0}\right).\end{aligned}\tag{18}$$

Eq. (17) represents the time behavior of the main equation (5) for the first seven modes with small bending stiffness terms. The second sum in Eq. (17) represents the influence of the beam with string effect model (i.e. the influence of the higher order modes ($k\geqslant 8$) on the lower order modes ($k<8$)). In the first sum in Eq. (18) there still is the influence of the string modes on the higher order modes. So, there is an interaction between the two models. In Eq. (18) the bending stiffness terms are now of leading order. The solution of the $\mathcal{O}(1)^{(1)}$-problem is given by

$$v_{k0}^{(1)}=A_{k0}(t_1)\sin(\Omega_k^{(1)}t_0)+B_{k0}(t_1)\cos(\Omega_k^{(1)}t_0),\quad k=1,2,\dots,7.\tag{19}$$

The solution of the $\mathcal{O}(1)^{(2)}$-problem is given by

$$v_{k0}^{(2)}=A_{k0}(t_1)\sin(\Omega_k^{(2)}t_0)+B_{k0}(t_1)\cos(\Omega_k^{(2)}t_0),\quad k=8,9,\dots\ .\tag{20}$$

In Eqs. (19) and (20) $\Omega_k^{(1)}$ and $\Omega_k^{(2)}$ are given by Eq. (11). $A_{k0}(t_1)$ and $B_{k0}(t_1)$ are still arbitrary functions and can be used to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$-problem and the $\mathcal{O}(\varepsilon)^{(2)}$-problem. The $\mathcal{O}(\varepsilon)^{(1)}$ equation now becomes (for $k=1,2,\dots,7$)

$$\begin{aligned}\frac{\partial^2 v_{k1}^{(1)}}{\partial t_0^2}+(\Omega_k^{(1)})^2v_{k1}^{(1)}=&-2\Omega_k^{(1)}\left(\frac{\partial A_{k0}}{\partial t_1}\cos(\Omega_k^{(1)}t_0)-\frac{\partial B_{k0}}{\partial t_1}\sin(\Omega_k^{(1)}t_0)\right)\\&-c_1k^4(A_{k0}(t_1)\sin(\Omega_k^{(1)}t_0)+B_{k0}(t_1)\cos(\Omega_k^{(1)}t_0))\\&+\sum_{n=1}^{7*}\left\{\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)A_{n0}(t_1)\sin(\Omega_n^{(1)}t_0)+B_{n0}(t_1)\cos(\Omega_n^{(1)}t_0)\right.\\&\left.+8(V_0+\alpha\sin(\omega t))\Omega_n^{(1)}(A_{n0}(t_1)\cos(\Omega_n^{(1)}t_0)-B_{n0}(t_1)\sin(\Omega_n^{(1)}t_0))\right\}\\&+\sum_{n=8}^{\infty*}\left\{\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)A_{n0}(t_1)\sin(\Omega_n^{(2)}t_0)+B_{n0}(t_1)\cos(\Omega_n^{(2)}t_0)\right.\\&\left.+8(V_0+\alpha\sin(\omega t))\Omega_n^{(2)}(A_{n0}(t_1)\cos(\Omega_n^{(2)}t_0)-B_{n0}(t_1)\sin(\Omega_n^{(2)}t_0))\right\},\end{aligned}\tag{21}$$

and the $\mathcal{O}(\varepsilon)^{(2)}$ equation is given by (for $k = 8, 9, \ldots$)

$$\begin{aligned}\frac{\partial^2 v_{k1}^{(2)}}{\partial t_0^2} + (\Omega_k^{(2)})^2 v_{k1}^{(2)} = & -2\Omega_k^{(2)}\left(\frac{\partial A_{k0}}{\partial t_1}\cos(\Omega_k^{(2)} t_0) - \frac{\partial B_{k0}}{\partial t_1}\sin(\Omega_k^{(2)} t_0)\right) \\ & + \sum_{n=1}^{7*}\left\{\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)A_{n0}(t_1)\sin(\Omega_n^{(1)} t_0) + B_{n0}(t_1)\cos(\Omega_n^{(1)} t_0)\right. \\ & \left. + 8(V_0 + \alpha\sin(\omega t))\Omega_n^{(1)}(A_{n0}(t_1)\cos(\Omega_n^{(1)} t_0) - B_{n0}(t_1)\sin(\Omega_n^{(1)} t_0))\right\} \\ & + \sum_{n=8}^{\infty*}\left\{\frac{kn}{(n^2-k^2)\pi}4\omega\alpha\cos(\omega t)A_{n0}(t_1)\sin(\Omega_n^{(2)} t_0) + B_{n0}(t_1)\cos(\Omega_n^{(2)} t_0)\right. \\ & \left. + 8(V_0 + \alpha\sin(\omega t))\Omega_n^{(2)}\left(A_{n0}(t_1)\cos(\Omega_n^{(2)} t_0) - B_{n0}(t_1)\sin(\Omega_n^{(2)} t_0)\right)\right\}.\end{aligned} \tag{22}$$

From Eqs. (21) and (22) it can readily be seen that there are infinitely many values of ω that can give rise to internal resonances. In fact these values are (in an $\mathcal{O}(\varepsilon)$ neighborhood of)

$$\text{(i) } \omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(1)} \quad \text{for } n, k = 1, 2, \ldots, 7,$$

$$\text{(ii) } \omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(1)} \quad \text{for } k = 1, 2, \ldots, 7, \text{ and } n = 8, 9, \ldots,$$

$$\text{(iii) } \omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(2)} \quad \text{for } n = 1, 2, \ldots, 7, \text{ and } k = 8, 9, \ldots,$$

$$\text{(iv) } \omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(2)} \quad \text{for } n = 8, 9, \ldots, \text{ and } k = 8, 9, \ldots . \tag{23}$$

For all resonant cases (i)–(iv) the additional condition that $k \pm n$ is an odd number, still holds due to summation in Eq. (15). By interchanging n and k, the resonant case (ii) (derived out of the $\mathcal{O}(\varepsilon)^{(1)}$-problem) becomes the resonant case (iii) (derived out of the $\mathcal{O}(\varepsilon)^{(2)}$-problem). The resonant case (i) is a resonance condition for the string equation, and has been investigated in Ref. [1]. The resonant case (iv) is a resonance condition for the beam with string effect equation. The solutions and stability conditions for this case can be found in Ref. [3]. Due to the interactions of these two models the model as proposed here has additional resonance conditions (ii) and (iii), where ω might be the sum or difference of one natural frequency of the string and one natural frequency of the beam with string effect. It is also necessary to investigate additionally if ω in the resonant case (i) also satisfies the cases (ii), (iii), and (iv), and vice versa. In the following section μ and ε will be taken equal to 0.002 and 0.01, respectively, and the initial value problem for Eq. (15) will be studied for different values of the relative error (in the frequencies) and for different values of ω.

4. Application of the method with a relative error of 5%

In this section the non-resonant and some resonant cases will be studied when the relative error in the frequencies is at most 5%.

4.1. The non-resonant case

In this case it is assumed that the frequency ω of the velocity-fluctuations of the axially moving continuum is not equal to any combination of the resonance frequencies as listed in Eq. (23). To eliminate the secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$-problem and the $\mathcal{O}(\varepsilon)^{(2)}$-problem it follows that A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \dfrac{dA_{k0}}{dt_1} = -\dfrac{c_1 k^3}{2} B_{k0}, \\ \dfrac{dB_{k0}}{dt_1} = \dfrac{c_1 k^3}{2} A_{k0}, \end{cases} \tag{24}$$

for $1 \leqslant k \leqslant 7$, and where c_1 is given by $\mu = c_1 \varepsilon$, and

$$\begin{cases} \dfrac{\mathrm{d}A_{k0}}{\mathrm{d}t_1} = 0 \\ \dfrac{\mathrm{d}B_{k0}}{\mathrm{d}t_1} = 0 \end{cases} \Longleftrightarrow \begin{cases} A_{k0}(t_1) = A_{k0}(0), \\ B_{k0}(t_1) = B_{k0}(0), \end{cases} \tag{25}$$

for $8 \leqslant k < \infty$. In this case system (24) can be seen as some sort of correction on the slow time (t_1) behavior of the solution in the first seven vibration modes due to the presence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$-problem. In fact it can be seen as a correction on the frequencies of the oscillation modes for $1 \leqslant k \leqslant 7$ since the solution of Eq. (24) is given by

$$\begin{cases} A_{k0}(t_1) = K_{1k} \cos\left(\dfrac{c_1 k^3}{2} t_1\right) - K_{2k} \sin\left(\dfrac{c_1 k^3}{2} t_1\right), \\ B_{k0}(t_1) = K_{1k} \sin\left(\dfrac{c_1 k^3}{2} t_1\right) + K_{2k} \cos\left(\dfrac{c_1 k^3}{2} t_1\right), \end{cases} \tag{26}$$

for $1 \leqslant k \leqslant 7$, and where K_{1k} and K_{2k} are all constant of integration. From the initial conditions (7) it follows that

$$f(x) = \sum_{k=1}^{\infty} u_k(0;\varepsilon)\sin(kx) \Longleftrightarrow u_k(0;\varepsilon) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,\mathrm{d}x,$$

$$r(x) = \sum_{k=1}^{\infty} \dot{u}_k(0;\varepsilon)\sin(kx) \Longleftrightarrow \dot{u}_k(0;\varepsilon) = \frac{2}{\pi}\int_0^{\pi} r(x)\sin(kx)\,\mathrm{d}x.$$

Moreover, since $u_k(0;\varepsilon) = v_k(0,0;\varepsilon) = v_{k0}(0,0;\varepsilon) + \varepsilon v_{k1}(0,0;\varepsilon) + \varepsilon^2 v_{k2}(0,0;\varepsilon) + \cdots$, and $\dot{u}_k(0;\varepsilon) = \dot{v}_k(0,0;\varepsilon) = \dot{v}_{k0}(0,0;\varepsilon) + \varepsilon\dot{v}_{k1}(0,0;\varepsilon) + \varepsilon^2\dot{v}_{k2}(0,0;\varepsilon) + \cdots$ it follows that

$$v_k(0;\varepsilon) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,\mathrm{d}x \quad \text{and} \quad \dot{v}_k(0;\varepsilon) = \frac{2}{\pi}\int_0^{\pi} r(x)\sin(kx)\,\mathrm{d}x. \tag{27}$$

From Eqs. (19), (20) and (27) the following condition for $A_{k0}(0)$ and $B_{k0}(0)$ can be obtained:

$$A_{k0}(0) = \frac{2k}{\pi}\int_0^{\pi} r(x)\sin(kx)\,\mathrm{d}x,$$

$$B_{k0}(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,\mathrm{d}x. \tag{28}$$

All constants of integration in Eq. (26) can now be found using Eq. (28). The solution $v_{k0}^{(1)}$ can now be obtained for $1 \leqslant k \leqslant 7$, that is

$$v_{k0}^{(1)} = A_{k0}(0)\sin(k(t_0 + 0.1k^3 t_1)) + B_{k0}(0)\cos(k(t_0 + 0.1k^3 t_1)). \tag{29}$$

The solution $v_{k0}^{(2)}$ for $k = 8, 9, \ldots$ can now also be found, using Eqs. (20) and (25):

$$v_{k0}^{(2)} = A_{k0}(0)\sin(k\sqrt{1 + 0.002k^2}\,t_0) + B_{k0}(0)\cos(k\sqrt{1 + 0.002k^2}\,t_0). \tag{30}$$

From Eq. (29) it follows that $k(t_0 + 0.1k^3 t_1) = (k + 0.001k^3)t$, so the frequencies for the first seven modes are approximated by $k + 0.001k^3$. Observing Eq. (30) the frequency of the higher modes is $k\sqrt{1 + 0.002k^2}$. The exact frequencies for $k = 1, \ldots, 7$ are also given by $k\sqrt{1 + 0.002k^2}$. The difference between the exact and the approximated frequencies are $|k + 0.001k^3 - k\sqrt{1 + 0.002k^2}| \leqslant 0.008$ for $1 \leqslant k \leqslant 7$. This means that due to the application of the two time-scales perturbation method there is a slow time t_1 correction in the string model frequency which represents the effect of the small bending stiffness for the lower oscillation modes with $1 \leqslant k \leqslant 7$.

4.2. Some resonant cases

The following resonant cases will be investigated:

$$\omega = m^* \quad \text{(where } m^* \text{ is equal to } 1, 3, 5 \text{ or } 7\text{)},$$

$$\omega = \Omega_9^{(2)} - \Omega_8^{(2)} \quad \text{(a difference type of resonance)},$$

$$\omega = \Omega_1^{(1)} - \Omega_8^{(2)} \quad \text{(an additional difference type of resonance)},$$

$$\omega = \Omega_9^{(2)} + \Omega_8^{(2)} \quad \text{(a sum type of resonance),}$$

$$\omega = \Omega_1^{(1)} + \Omega_8^{(2)} \quad \text{(an additional sum type of resonance),}$$

$$\omega = \Omega_7^{(1)} - \Omega_8^{(2)} \quad \text{(an additional difference type of resonance),}$$

$$\omega = \Omega_7^{(1)} + \Omega_8^{(2)} \quad \text{(an additional sum type of resonance), and}$$

$$\omega = 2\Omega_8^{(2)} \quad \text{(usually referred to as a principal parametric resonance).} \tag{31}$$

4.2.1. The resonant case $\omega = m^*$, where m^* is equal to 1, 3, 5 or 7

First, the case $\omega = m^* = 1$ (the first "string" resonant frequency) will be studied. It was shown in Ref. [1] that the case (i) in Eq. (23), i.e. $\omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(1)}$, with $n, k = 1, 2, \ldots, 7$, and $\omega = 1$ has solutions: $k = n+1$ and $k = n-1$. Additionally it has to be checked whether the resonance conditions (ii)–(iv) in Eq. (23) also have solutions if $\omega = 1$ or not. It can readily be verified that there are no such k and n. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$-problem (see Eq. (21)) A_{k0} and B_{k0} for $1 \leqslant k \leqslant 7$ have to satisfy

$$\begin{cases} \dfrac{\mathrm{d}A_{k0}}{\mathrm{d}t_1} = -\dfrac{c_1 k^3}{2} B_{k0} - \dfrac{\alpha}{\pi}(k-1)B_{(k-1)0} - \dfrac{\alpha}{\pi}(k+1)B_{(k+1)0}, \\ \dfrac{\mathrm{d}B_{k0}}{\mathrm{d}t_1} = \dfrac{c_1 k^3}{2} A_{k0} + \dfrac{\alpha}{\pi}(k-1)A_{(k-1)0} + \dfrac{\alpha}{\pi}(k+1)A_{(k+1)0}, \end{cases} \tag{32}$$

and the other A_{k0} and B_{k0} functions with $8 \leqslant k < \infty$, derived from the $\mathcal{O}(\varepsilon)^{(2)}$-problem (22), satisfy the same equations as for the non-resonant case (see Eq. (25)). It is clear from system (32), and from system (25), that there are interactions between the first eight modes. For the higher order vibration modes (i.e. for the 9th vibration mode and higher) there are no such interactions. Therefore there is no problem with applying the truncation method as it was in Ref. [1] for a string model (without bending stiffness). For the case $\omega = m^*$ it is assumed that m^* is equal to 3, 5, or 7 ($\omega = m^*$ is a resonance frequency from the string model). Following the same procedure as for the case $\omega = 1$, the following equations for A_{k0} and B_{k0} can be found:

$$\begin{cases} \dfrac{\mathrm{d}A_{k0}}{\mathrm{d}t_1} = -\dfrac{c_1 k^3}{2m^*} B_{k0} - \dfrac{\alpha(k-m^*)(2k-2m^*+1)}{\pi \quad m^*(2k-m^*)} B_{(k-m^*)0} - \dfrac{\alpha(k+m^*)(2k+2m^*-1)}{\pi \quad m^*(2k+m^*)} B_{(k+m^*)0}, \\ \dfrac{\mathrm{d}B_{k0}}{\mathrm{d}t_1} = \dfrac{c_1 k^3}{2m^*} A_{k0} + \dfrac{\alpha(k-m^*)(2k-2m^*+1)}{\pi \quad m^*(2k-m^*)} A_{(k-m^*)0} + \dfrac{\alpha(k+m^*)(2k+2m^*-1)}{\pi \quad m^*(2k+m^*)} A_{(k+m^*)0}, \end{cases} \tag{33}$$

for $1 \leqslant k \leqslant 7$ (where A_{k0} and B_{k0} are assumed to be zero for $k \leqslant 0$). For the higher order modes ($8 \leqslant k < \infty$) system (25) still holds. It can be seen from Eqs. (33) and (25) that there are interactions between the first $(7 + m^*)$ modes. For the higher order modes there are no interactions.

To check the stability of the solution the following approach can be used. From Eqs. (32) and (33) it follows that, in general, the equation $\dot{X} = MX$ has to be solved, where the vector X contains the unknown functions $A_{k0}(t_1)$ and $B_{k0}(t_1)$, and where M is a corresponding square matrix of the size (7+m^*) with constant elements. The eigenvalues of the matrix M define the stability properties of the system. It turns out that for all m^* (that is, for $m^* = 1, 3, 5,$ and 7) all eigenvalues of matrix M are purely imaginary or zeros, and that all solutions are bounded and stable in this case.

4.2.2. The resonant case $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$: a difference type of resonance

One of the difference type of resonances of the stretched beam model (see case (iv) of Eq. (23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_9^{(2)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(2)}$ for $n = 8, 9, \ldots$ and $k = 8, 9, \ldots$ has only trivial solutions for $n = 8$ and $k = 9$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$, and $n = 9$ and $k = 8$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ (detailed calculations to prove this can be found in Ref. [3]). Additionally it has to be checked whether the resonance conditions (i)–(iii) in Eq. (23) also have solutions if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ or not. It can be verified that there are no such k and n. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(2)}$-problem (see Eq. (22)) A_{k0} and B_{k0} have to satisfy:

$$\begin{cases} \dfrac{\mathrm{d}A_{8,0}}{\mathrm{d}t_1} = -\dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_9^{(2)} + \Omega_8^{(2)}}{\Omega_8^{(2)}}\right) B_{9,0}, \\ \dfrac{\mathrm{d}B_{8,0}}{\mathrm{d}t_1} = \dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_9^{(2)} + \Omega_8^{(2)}}{\Omega_8^{(2)}}\right) A_{9,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{9,0}}{dt_1} = -\dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_9^{(2)}+\Omega_8^{(2)}}{\Omega_9^{(2)}}\right)B_{8,0}, \\ \dfrac{dB_{9,0}}{dt_1} = \dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_9^{(2)}+\Omega_8^{(2)}}{\Omega_9^{(2)}}\right)A_{8,0}. \end{cases} \tag{34}$$

It can be seen from Eq. (34) that there are interactions between the 8th and the 9th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$ -problems if $\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$ the following results were found. For vibration modes with $1 \leqslant k \leqslant 7$ A_{k0} and B_{k0} are still given by Eq. (24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$-problem. For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy Eq. (34). And for the higher order vibration modes with $10 \leqslant k < \infty$ A_{k0} and B_{k0} have to satisfy Eq. (25). To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

4.2.3. The resonant case $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

One of the additional difference type of resonances of the string–beam model (see case (ii) of Eq. (23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(1)}$ for $n = 8, 9, \ldots$ and $k = 1, 2, \ldots 7$ has only a trivial solution for $n = 8$ and $k = 1$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in Eq. (23) also have solutions if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see Eq. (23)): the equation $\omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(2)}$ for $k = 8, 9, \ldots$ and $n = 1, 2, \ldots 7$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems (see Eqs. (21) and (22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \dfrac{dA_{1,0}}{dt_1} = -\dfrac{c_1}{2\Omega_1^{(1)}}B_{1,0} + \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)}+\Omega_8^{(2)}}{\Omega_1^{(1)}}\right)B_{8,0}, \\ \dfrac{dB_{1,0}}{dt_1} = \dfrac{c_1}{2\Omega_1^{(1)}}A_{1,0} - \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)}+\Omega_8^{(2)}}{\Omega_1^{(1)}}\right)A_{8,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{8,0}}{dt_1} = \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)}+\Omega_8^{(2)}}{\Omega_8^{(2)}}\right)B_{1,0}, \\ \dfrac{dB_{8,0}}{dt_1} = -\dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)}+\Omega_8^{(2)}}{\Omega_8^{(2)}}\right)A_{1,0}. \end{cases} \tag{35}$$

It can be seen from Eq. (35) that there are interactions between the 1st and the 8th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems if $\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of Eq. (35). For the vibration modes with $2 \leqslant k \leqslant 7$ A_{k0} and B_{k0} are still given by Eq. (24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} are given by the second system of Eq. (35). And for the higher order vibration modes with $9 \leqslant k < \infty$ A_{k0} and B_{k0} have to satisfy Eq. (25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

4.2.4. The resonant case $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$: a sum type of resonance

One of the sum type of resonances of the stretched beam model (see case (iv) of Eq. (23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is equal to $\Omega_9^{(2)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(2)}$ for $n = 8, 9, \ldots$ and $k = 8, 9, \ldots$ has only trivial solutions for $n = 8$ and $k = 9$, and symmetrically $n = 9$ and $k = 8$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(2)}$ (detailed calculations to prove this can be found in Ref. [3]). Additionally it has to be checked whether the resonance conditions (i)–(iii) in Eq. (23) also have solutions or not if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$. It can be verified that there are no such k and n. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(2)}$-problem (see Eq. (22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \dfrac{dA_{8,0}}{dt_1} = \dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_8^{(2)}}\right)B_{9,0}, \\ \dfrac{dB_{8,0}}{dt_1} = \dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_8^{(2)}}\right)A_{9,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{9,0}}{dt_1} = \dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_9^{(2)}}\right)B_{8,0}, \\ \dfrac{dB_{9,0}}{dt_1} = \dfrac{72\alpha}{17\pi}\left(\dfrac{\Omega_8^{(2)} - \Omega_9^{(2)}}{\Omega_9^{(2)}}\right)A_{8,0}. \end{cases} \tag{36}$$

It can be seen from Eq. (36) that there are interactions between the 8th and the 9th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems if $\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$ the following results were found. For the vibration modes with $1 \leqslant k \leqslant 7$ A_{k0} and B_{k0} still have to satisfy Eq. (24). This can be seen as an influence of the small bending stiffness term in the $\mathcal{O}(\varepsilon)^{(1)}$-problem. For the 8th and the 9th vibration modes A_{k0} and B_{k0} have to satisfy Eq. (36). And for the higher order vibration modes with $10 \leqslant k < \infty$ A_{k0} and B_{k0} have to satisfy Eq. (25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

4.2.5. The resonant case $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

One of the additional sum type of resonances of the string–beam model (see case (ii) of Eq. (23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_1^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(1)}$ for $n = 8, 9, \ldots$ and $k = 1, 2, \ldots 7$ has only a trivial solution for $n = 8$ and $k = 1$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in Eq. (23) also have solutions if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) case (see Eq. (23)): the equation $\omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(2)}$ for $k = 8, 9, \ldots$ and $n = 1, 2, \ldots 7$ has a solution $n = 1$ and $k = 8$ if $\omega - \Omega_n^{(1)} = \Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems (see Eqs. (21) and (22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \dfrac{dA_{1,0}}{dt_1} = -\dfrac{c_1}{2\Omega_1^{(1)}}B_{1,0} + \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_1^{(1)}}\right)B_{8,0}, \\ \dfrac{dB_{1,0}}{dt_1} = \dfrac{c_1}{2\Omega_1^{(1)}}A_{1,0} + \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_1^{(1)}}\right)A_{8,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{8,0}}{dt_1} = \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}}\right)B_{1,0}, \\ \dfrac{dB_{8,0}}{dt_1} = \dfrac{8\alpha}{63\pi}\left(\dfrac{\Omega_1^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}}\right)A_{1,0}. \end{cases} \tag{37}$$

It can be seen from Eqs. (37) that there are interactions between the 1st and the 8th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems if $\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$ the following results were found. For the 1st vibration mode (with $k = 1$) A_{k0} and B_{k0} are given by the first system of Eq. (37). For the vibration modes with $2 \leqslant k \leqslant 7$ A_{k0} and B_{k0} still have to satisfy Eq. (24). For the 8th vibration mode (with $k = 8$) A_{k0} and B_{k0} have to satisfy the second system of Eq. (37). And for the higher order vibration modes with $9 \leqslant k < \infty$ A_{k0} and B_{k0} have to satisfy Eq. (25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

4.2.6. The resonant case $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$: an additional difference type of resonance

One of the additional difference type of resonances of the string–beam model (see case (ii) of Eq. (23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} - \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(1)}$ for $n = 8, 9, \ldots$ and $k = 1, 2, \ldots 7$ has only a trivial solution $n = 8$ and $k = 7$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in Eq. (23) also have solutions or not if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$. It turns out that there is one solution in case (iii) (see Eq. (23)): the equation $\omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(2)}$ for $k = 8, 9, \ldots$ and $n = 1, 2, \ldots 7$ has a solution $n = 7$ and $k = 8$ if $\omega - \Omega_n^{(1)} = -\Omega_k^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems (see Eqs. (21) and (22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \dfrac{dA_{7,0}}{dt_1} = -\dfrac{c_1 2401}{2\Omega_7^{(1)}} B_{7,0} + \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_7^{(1)}}\right) B_{8,0}, \\ \dfrac{dB_{7,0}}{dt_1} = \dfrac{c_1 2401}{2\Omega_7^{(1)}} A_{7,0} - \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_7^{(1)}}\right) A_{8,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{8,0}}{dt_1} = \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_8^{(2)}}\right) B_{7,0}, \\ \dfrac{dB_{8,0}}{dt_1} = -\dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} + \Omega_8^{(2)}}{\Omega_8^{(2)}}\right) A_{7,0}. \end{cases} \tag{38}$$

It can be seen from Eq. (35) that there are interactions between the 7th and the 8th vibration modes. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems if $\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$ the following results were found. For the vibration modes with $1 \leqslant k \leqslant 6$ A_{k0} and B_{k0} still have to satisfy Eq. (24). For the 7th and 8th vibration modes (with $k = 7$ and 8) A_{k0} and B_{k0} have to satisfy Eq. (38). And for the higher order vibration modes with $9 \leqslant k < \infty$ A_{k0} and B_{k0} have to satisfy Eq. (25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

4.2.7. The resonant case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$: an additional sum type of resonance

One of the additional sum type of resonances of the string–beam model (see case (ii) of Eq. (23)) will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $\Omega_7^{(1)} + \Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(1)}$ for $n = 8, 9, \ldots$ and $k = 1, 2, \ldots 7$ has only a trivial solution $n = 8$ and $k = 7$ if $\omega - \Omega_n^{(2)} = \Omega_k^{(1)}$. Additionally it has to be checked whether the resonance conditions (i), (iii) and (iv) in Eq. (23) also have any solutions if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ or not. It turns out that there is one solution in case (iii) (see Eq. (23)): the equation $\omega \pm \Omega_n^{(1)} = \pm\Omega_k^{(2)}$ for $k = 8, 9, \ldots$ and $n = 1, 2, \ldots 7$ has a solution $n = 7$ and $k = 8$ if $\omega - \Omega_n^{(1)} = \Omega_k^{(2)}$. It was also found for the case (iv) of Eq. (23) that $n = 20$ and $k = 27$ if $\omega + \Omega_n^{(2)} = \Omega_k^{(2)}$ and $n = 27$ and $k = 20$ if $\omega - \Omega_n^{(2)} = -\Omega_k^{(2)}$ then $|\omega \pm \Omega_n^{(2)} \pm \Omega_k^{(2)}| = 0.0012 < \varepsilon$. This can be seen as a detuning case if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems (see Eqs. (21) and (22)) A_{k0} and B_{k0} have to satisfy

$$\begin{cases} \dfrac{dA_{7,0}}{dt_1} = -\dfrac{c_1 2401}{2\Omega_7^{(1)}} B_{7,0} + \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_7^{(1)}}\right) B_{8,0}, \\ \dfrac{dB_{7,0}}{dt_1} = \dfrac{c_1 2401}{2\Omega_7^{(1)}} A_{7,0} + \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_7^{(1)}}\right) A_{8,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{8,0}}{dt_1} = \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}}\right) B_{7,0}, \\ \dfrac{dB_{8,0}}{dt_1} = \dfrac{56\alpha}{15\pi}\left(\dfrac{\Omega_7^{(1)} - \Omega_8^{(2)}}{\Omega_8^{(2)}}\right) A_{7,0}, \end{cases} \tag{39}$$

and, additionally,

$$\begin{cases} \dfrac{dA_{20,0}}{dt_1} = -\dfrac{540\alpha}{329\pi}\left(\dfrac{\Omega_{27}^{(2)}+\Omega_{20}^{(2)}}{\Omega_{20}^{(2)}}\right)B_{27,0}, \\ \dfrac{dB_{20,0}}{dt_1} = \dfrac{540\alpha}{329\pi}\left(\dfrac{\Omega_{27}^{(2)}+\Omega_{20}^{(2)}}{\Omega_{20}^{(2)}}\right)A_{27,0} \end{cases}$$

and

$$\begin{cases} \dfrac{dA_{27,0}}{dt_1} = -\dfrac{540\alpha}{329\pi}\left(\dfrac{\Omega_{27}^{(2)}+\Omega_{20}^{(2)}}{\Omega_{27}^{(2)}}\right)B_{20,0}, \\ \dfrac{dB_{27,0}}{dt_1} = \dfrac{540\alpha}{329\pi}\left(\dfrac{\Omega_{27}^{(2)}+\Omega_{20}^{(2)}}{\Omega_{27}^{(2)}}\right)A_{20,0}. \end{cases} \tag{40}$$

It can be seen from Eqs. (39) and (40) that there are interactions between the 7th and the 8th, and the 20th and the 27th vibration modes, respectively. In order to avoid secular terms in the solutions of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(2)}$-problems if $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ the following results were found. For the vibration modes with $1 \leqslant k \leqslant 6$ A_{k0} and B_{k0} are still given by Eq. (24). For the 7th and 8th vibration modes (with $k = 7$ and 8) A_{k0} and B_{k0} are given by Eq. (39). For the vibration modes with $9 \leqslant k \leqslant 19$ and $21 \leqslant k \leqslant 26$ A_{k0} and B_{k0} still have to satisfy Eq. (25). For the 20th and 27th vibration modes (with $k = 20$ and 27) A_{k0} and B_{k0} have to satisfy Eq. (40). And for the higher order vibration modes with $28 \leqslant k < \infty$ A_{k0} and B_{k0} have to satisfy Eq. (25).

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that some of the eigenvalues have positive real parts, and that some of the solutions in this case are unstable.

4.2.8. *The resonant case $\omega = 2\Omega_8^{(2)}$: a principal parametric resonance*

One of the principal parametric resonances of the stretched beam model will be studied in this subsection. The frequency of the belt velocity fluctuations ω is assumed to be equal to $2\Omega_8^{(2)}$. The equation $\omega \pm \Omega_n^{(2)} = \pm\Omega_k^{(2)}$ for $n = 8, 9, \ldots$ and $k = 8, 9, \ldots$ has no solutions. Additionally it has to be checked whether the resonance conditions (i)–(iii) in Eq. (23) also have solutions or not if $\omega = 2\Omega_8^{(2)}$. It can be verified that there are no such k and n. Therefore, in order to eliminate secular terms in the solution of the $\mathcal{O}(\varepsilon)^{(1)}$ and $\mathcal{O}(\varepsilon)^{(1)}$-problems (see Eqs. (21) and (22)) A_{k0} and B_{k0} have to satisfy (the same as for the non-resonant case): Eq. (32) for $1 \leqslant k \leqslant 7$ and Eq. (33) for $8 \leqslant k < \infty$. It has already been shown in Ref. [3] that in case of principal parametric resonance the possibility to have solutions (interactions between vibration modes) for the case (iv) of Eq. (23) depends on the value of μ. It turns out here that in the particular case $\mu = 0.002$ (as it was fixed in the beginning) $\omega = 2\Omega_8^{(2)}$ is not a resonance frequency.

To check the stability the eigenvalues of the corresponding matrix M have to be calculated. It turns out that all eigenvalues are purely imaginary or zeros, and that all solutions in this case are bounded and stable.

5. Application of the method with a relative error of 3%, 1%, and 0.1%

The analysis as given in the previous section (when the relative errors in the frequencies are at most 5%) can be repeated for the cases when the relative errors in the frequencies are at most 3%, or 1%, or 0.1%. The detailed computations will be

Table 2
Stability properties of the solutions.

Frequency ω	Relative error				
	5%	3%	1%	0.1%	Exact 0% (see Ref. [3])
$\omega = m^*$, m^* =1,3,5 and 7	Stable	Stable	Stable	Stable	Stable
$\omega = \Omega_9^{(2)} - \Omega_8^{(2)}$	Stable	Stable	Stable	Stable	Stable
$\omega = \Omega_1^{(1)} - \Omega_8^{(2)}$	Stable	Stable	Stable	Stable	Stable
$\omega = \Omega_9^{(2)} + \Omega_8^{(2)}$	Unstable	Unstable	Unstable	Unstable	Unstable
$\omega = \Omega_1^{(1)} + \Omega_8^{(2)}$	Unstable	Unstable	Unstable	Unstable	Unstable
$\omega = \Omega_7^{(1)} - \Omega_8^{(2)}$	Stable	Stable	Stable	Stable	Stable
$\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$	Unstable	Stable	Stable	Stable	Stable
$\omega = 2\Omega_8^{(2)}$	Stable	Stable	Stable	Stable	Stable

omitted in this paper, but can be found in Ref. [15]. The stability properties of the solutions when the relative errors in the frequencies are at most 5%, 3%, 1%, or 0.1%, as well as the exact stability properties (i.e. the relative error is 0%) are given in Table 2.

To verify the method which was proposed in this paper the following approach will be used. Stability properties of the solutions which were derived will be compared with the stability properties of the exact solution of problem (5) (the correct properties can be found in Ref. [3]). It can be seen from Table 2 that the stability properties remain the same for almost all these cases, and correspond to the stability properties of the exact solution of the problem. For the case $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ (see Table 2) and relative error of at most 5% the solution is unstable. On the other hand, for the same frequency ω but with a better accuracy (relative error of 3%, 1%, and 0.1%) the solution is stable, which corresponds also to the exact solution of the problem. This difference occurs because the resonance frequency $\omega = \Omega_7^{(1)} + \Omega_8^{(2)}$ was derived from the combination of the string model and the stretched beam model. For the case of a relative error of 5% (see Section 4) the value $\Omega_7^{(1)}$ is still in the region of the string model. For the other cases the region of the string model is smaller, so that this combination is not a resonance frequency anymore. It can also be seen from Table 2 that difference type of resonances are stable and sum type or resonances are unstable. This behavior corresponds with the stability properties of the exact solution which was found in Ref. [3].

6. Conclusions and remarks

In this paper an initial-boundary value problem for a linear equation, describing an axially moving stretched beam has been studied. This equation can be used as a model for the transversal vibrations of a conveyor belt system. The axially moving belt is assumed to move in one direction with a non-constant speed $V(t)$, that is, $V(t) = \varepsilon(V_0 + \alpha \sin(\omega t))$, where $0 < \varepsilon \ll 1$, and where V_0, α and ω are positive constants. For V_0 it is assumed that $V_0 > 0$ and $V_0 > |\alpha|$. A new model approach describing the transient "from string to beam" behavior, based on the calculations of the natural frequencies has been proposed. The influence of the bending stiffness on the stability properties of the solution of the problem has been studied.

Depending on the natural frequencies the original problem is split up into two sub-models: a string model for the lower frequencies and a stretched beam model for the higher frequencies. Each sub-model has its own physical and mathematical properties. For the string model, for instance, the discussion on the applicability of the truncation method is not (mathematically) relevant anymore (see also Ref. [1]). In this combination model the sub-models are interacting due to internal resonances, and the model equations depend on the frequencies and on the vibration mode numbers. The proposed model is a more realistic approach to describe the dynamical behavior of a traveling continuum as the bending stiffness becomes more important for the higher order vibration modes. For the lower frequencies the bending stiffness can be neglected and the string equation can be used. The regions of applicability of the simplified models were found for different values of the bending stiffness parameter and for different values of the relative errors in the frequencies. It turns out that there are infinitely many values of ω that give rise to internal resonances in the axially moving belt system. In fact, that happens when ω is equal to any sum or difference combination of the natural frequencies of the string and (or) the stretched beam equations. In the non-resonant case it can clearly be seen that for the lower frequencies (when the string model is used) the perturbation approach leads to improvements in the frequencies taking into account the small bending stiffness. The formal approximations of the solution and the stability properties in some resonant cases have been determined for four different values of the relative error in the frequencies: 5%, 3%, 1% and 0.1%. It can be concluded that the properties of the studied problem remain the same and correspond to the properties of the exact solution of the problem (5) when the relative errors in the frequencies are less than 5%.

An important implication of the results as presented in the literature (see, for instance Refs. [1–3]) is that for these types of problems the use of only string-like models is not appropriate. To describe the dynamics of these types of conveyor belt problems correctly one has to include bending stiffness in the model (also when the bending stiffness is assumed to be small). In this paper it has explicitly been shown how one should work with a combined model that is a string model at the low frequencies and a tensioned beam model at the higher frequencies.

It has also to be noticed that the introduction of a damping term does not solve the truncation problem for the string-like equation (1), at least if the damping is assumed to be small and of order ε. The viscous damping or the structural damping can be taken into account for the partial differential equations (1) and (2), leading to the appearance of extra terms in these equations: $u_t + Vu_x$, $u_{txx} + Vu_{xxx}$, or $u_{txxxx} + Vu_{xxxxx}$, respectively (for the moving frame of reference). If the damping in the problem is assumed to be small, that is of order ε, then after applying the two time-scales perturbation method to the equations, terms $u_t + Vu_x$, $u_{txx} + Vu_{xxx}$, or $u_{txxxx} + Vu_{xxxxx}$ will appear. At high frequencies the structural damping terms cannot be considered to be small. The solutions of the $\mathcal{O}(1)$-problems still cannot be truncated in these cases and still has to be written in an infinite series representation from the mathematical point of view.

For future research it will be interesting to study more complicated cases for the transversal and longitudinal motions of axially moving beams, including those cases for which the boundary conditions are such that energy inflow or outflow is possible through these boundaries. Numerical results (based on spectral element methods) for these type of problems can already be found in [16,17].

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